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THE DISTURBING FUNCTION FOR SOME OF THE HIGHER DEGREE TESSERAL HARMONICS

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NASA GODDARD SPACE FLIGHT CENTER
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SUMMARY

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The disturbing function due to certain tesseral harmonics (m,n) for (1,5), (2,5), (3,5), (4,5), (5,5), (1,6), (2,6), (3,6), (4,6), (5,6), (6,6), (1,7), (2,7), (3,7), (4,7), (5,7), (6,7), (13,13), (13, 15), and (14,15) is given both as a function of Keplerian elements of a satellite and as a function of the rectangular coordinates.

Author

THE DISTURBING FUNCTION FOR SOME OF THE HIGHER DEGREE TESSERAL HARMONICS

INTRODUCTION

The disturbing function for certain higher degree tesseral harmonics is given as a function of the Keplerian elements of the satellite and also in terms of the rectangular coordinates. The higher-degree tesseral harmonics (m,n) that may be of importance, based upon experimental evidence, are the following: (1,5), (2,5), (3,5), (4,5), (5,5), (1,6), (2,6), (3,6), (4,6), (5,6), (6,6), (1,7), (2,7), (3,7), (4,7), (5,7), (6,7), (13,13), (13,15), and (14,15).

THE FORCE FUNCTION

The force function U of the earth at a point whose spherical coordinates are (r, ϕ, λ) in an earth centered rotating coordinate system may be expressed as

$$U = \frac{\mu}{r}\left\{1 + \sum_{n=2}^{\infty}\sum_{m=0}^{n}\left(\frac{a_e}{r}\right)^n J_n^{(m)} P_n^{(m)}(\sin\phi)\cos m\left(\lambda - \lambda_n^{(m)}\right)\right\} \tag{1}$$

or

$$U = \frac{\mu}{r}\left\{1 + \sum_{n=2}^{\infty}\sum_{m=0}^{n}\left(\frac{a_e}{r}\right)^n P_n^{(m)}(\sin\phi)\left(C_n^{(m)}\cos m\lambda + S_n^{(m)}\sin m\lambda\right)\right\}, \tag{2}$$

where a_e is the earth's equatorial radius.

Here, only terms for which $m \neq 0$ are considered, and $P_n^{(m)}(\sin\phi)$ is the Legendre associated function defined by

$$P_n^{(m)}(x) = \frac{1}{2^n n!}(1 - x^2)^{\frac{m}{2}} \frac{d^{n+m}(x^2 - 1)^n}{dx^{n+m}}.$$

The relationships between the constants in the force function expressed in the different forms of Equation (1) and Equation (2) are

$$C_n^{(m)} = J_n^{(m)} \cos m\lambda_n^{(m)}$$

$$S_n^{(m)} = J_n^{(m)} \sin m\lambda_n^{(m)}.$$

For convenience, use is made of the normalized constants $\bar{S}_n^{(m)}$, $\bar{C}_n^{(m)}$, and $\bar{J}_n^{(m)}$ where

$$\bar{S}_n^{(m)} = N_n^{(m)} S_n^{(m)}$$

$$\bar{C}_n^{(m)} = N_n^{(m)} C_n^{(m)}$$

$$\bar{J}_n^{(m)} = N_n^{(m)} J_n^{(m)}$$

and where

$$N_n^{(m)} = \left[\frac{2(2n + 1)(n - m)!}{(n + m)!}\right]^{-\frac{1}{2}}. \qquad (m \neq 0)$$

The disturbing function for the tesseral and sectorial harmonics is then

$$R = \frac{\mu}{r} \sum_{n=2}^{\infty} \sum_{m=1}^{n} \left(\frac{a_e}{r}\right)^n \bar{J}_n^{(m)} \bar{P}_n^{(m)} (\sin \phi) \cos m\left(\lambda - \lambda_n^{(m)}\right)$$

or

$$R = \frac{\mu}{r} \sum_{n=2}^{\infty} \sum_{m=1}^{n} \left(\frac{a_e}{r}\right)^n \bar{P}_n^{(m)} (\sin \phi) \left(\bar{C}_n^{(m)} \cos m\lambda + \bar{S}_n^{(m)} \sin m\lambda\right).$$

THE DISTURBING FUNCTION AS A FUNCTION OF THE ORBITAL ELEMENTS

If one makes use of the following relationships between the spherical coordinates and the Keplerian elements (a, e, i, ℓ, g, h) and true anomaly, f, of the satellite,

$$\sin \phi = \sin i \sin(f + g)$$

$$\cos^m \phi \cos m\left(\lambda - \lambda_n^{(m)}\right) = \frac{1}{2^m}\left\{(1 + \cos i)^m \cos m\left(f + g + h - \theta_n^{(m)}\right)\right.$$

$$+ \binom{m}{1}(1 + \cos i)^{m-1}(1 - \cos i) \cos\left[(m - 2)(f + g) + m\left(h - \theta_n^{(m)}\right)\right]$$

$$+ \cdots\cdots$$

$$+ \binom{m}{m-1}(1 + \cos i)(1 - \cos i)^{m-1} \cos\left[(m-2)(f+g) - m\left(h - \theta_n^{(m)}\right)\right]$$

$$+ (1 - \cos i)^m \cos m\left(f + g - h + \theta_n^{(m)}\right)\Bigg\}$$

where $\theta_n^{(m)} = \lambda_n^{(m)} +$ Greenwich Sidereal Time, then the disturbing function for the harmonics being considered is

$$R = 5.0176690 \times 10^{-2} \frac{\mu \bar{J}_5^{(1)} a_e^5}{r^6} \Big\{2(1+c)(1+28c-42c^2-84c^3+105c^4)\cdot$$

$$\cdot \cos\left(f+g+h-\theta_5^{(1)}\right)$$

$$+ 2(1-c)(1-28c-42c^2+84c^3+105c^4)\cdot$$

$$\cdot \cos\left(f+g-h+\theta_5^{(1)}\right)$$

$$- 7s^2(1+c)(1+6c-15c^2) \cos\left(3f+3g+h-\theta_5^{(1)}\right)$$

$$- 7s^2(1-c)(1-6c-15c^2) \cos\left(3f+3g-h+\theta_5^{(1)}\right)$$

$$+ 21s^4(1+c) \cos\left(5f+5g+h-\theta_5^{(1)}\right)$$

$$+ 21s^4(1-c) \cos\left(5f+5g-h+\theta_5^{(1)}\right)\Big\}$$

$$+ .26551009 \frac{\mu \bar{J}_5^{(2)} a_e^5}{r^6} s\Big\{2(1+c)(1-3c-9c^2+15c^3) \sin\left(f+g+2h-2\theta_5^{(2)}\right)$$

$$+ 2(1-c)(1+3c-9c^2-15c^3) \sin\left(f+g-2h+2\theta_5^{(2)}\right)$$

$$- (1+c)^2(1-12c+15c^2) \sin\left(3f+3g+2h-2\theta_5^{(2)}\right)$$

$$- (1-c)^2(1+12c+15c^2) \sin\left(3f+3g-2h+2\theta_5^{(2)}\right)$$

$$- 3s^2(1+c)^2 \sin\left(5f+5g+2h-2\theta_5^{(2)}\right)$$

$$- 3s^2(1-c)^2 \sin\left(5f+5g-2h+2\theta_5^{(2)}\right)\Big\}$$

$$+ 5.4197019 \times 10^{-2} \frac{\mu \bar{J}_5^{(3)} a_e^5}{r^6} \Big\{ 6s^2(1+c)(1+6c-15c^2) \cos\left(f+g+3h-3\theta_5^{(3)}\right)$$

$$+ 6s^2(1-c)(1-6c-15c^2)\cos\left(f+g-3h+3\theta_5^{(3)}\right)$$

$$-(1+c)^3(13-54c+45c^2) \cos 3 \left(f+g+h-\theta_5^{(3)}\right)$$

$$-(1-c)^3(13+54c+45c^2) \cos 3 \left(f+g-h+\theta_5^{(3)}\right)$$

$$-9s^2(1+c)^3 \cos\left(5f+5g+3h-3\theta_5^{(3)}\right)$$

$$-9s^2(1-c)^3 \cos\left(5f+5g-3h+3\theta_5^{(3)}\right)\Big\}$$

$$+ .22993848 \frac{\mu \bar{J}_5^{(4)} a_e^5}{r^6} s \Big\{ 2s^2(1+c)(1-5c) \sin\left(f+g+4h-4\theta_5^{(4)}\right)$$

$$+ 2s^2(1-c)(1+5c)\sin\left(f+g-4h+4\theta_5^{(4)}\right)$$

$$+ (1+c)^3(3-5c) \sin\left(3f+3g+4h-4\theta_5^{(4)}\right)$$

$$+ (1-c)^3(3+5c) \sin\left(3f+3g-4h+4\theta_5^{(4)}\right)$$

$$+ (1+c)^4 \sin\left(5f+5g+4h-4\theta_5^{(4)}\right)$$

$$+ (1-c)^4 \sin\left(5f+5g-4h+4\theta_5^{(4)}\right)\Big\}$$

$$+7.2712932 \times 10^{-2} \frac{\mu \bar{J}_5^{(5)} a_e^5}{r^6} \Big\{ 10 s^4 (1+c) \cos\left(f + g + 5h - 5\theta_5^{(5)}\right)$$

$$+ 10 s^4 (1-c) \cos\left(f + g - 5h + 5\theta_5^{(5)}\right)$$

$$+ 5 s^2 (1+c)^3 \cos\left(3f + 3g + 5h - 5\theta_5^{(5)}\right)$$

$$+ 5 s^2 (1-c)^3 \cos\left(3f + 3g - 5h + 5\theta_5^{(5)}\right)$$

$$+ (1+c)^5 \cos 5\left(f + g + h - \theta_5^{(5)}\right)$$

$$+ (1-c)^5 \cos 5\left(f + g - h + \theta_5^{(5)}\right)\Big\}$$

$$+ 3.2270921 \times 10^{-2} \frac{\mu \bar{J}_6^{(1)} a_e^6}{r^7} \, s \Big\{ -20c \, (5 - 30c^2 + 33c^4) \sin\left(h - \theta_6^{(1)}\right)$$

$$+ 5(1+c)(1 + 18c - 36c^2 - 66c^3 + 99c^4) \cdot$$

$$\cdot \sin\left(2f + 2g + h - \theta_6^{(1)}\right)$$

$$+ 5(1-c)(1 - 18c - 36c^2 + 66c^3 + 99c^4) \cdot$$

$$\cdot \sin\left(2f + 2g - h + \theta_6^{(1)}\right)$$

$$- 6 s^2 (1+c)(2 + 11c - 33c^2) \sin\left(4f + 4g + h - \theta_6^{(1)}\right)$$

$$- 6 s^2 (1-c)(2 - 11c - 33c^2) \sin\left(4f + 4g - h + \theta_6^{(1)}\right)$$

$$+ 33 s^4 (1+c) \sin\left(6f + 6g + h - \theta_6^{(1)}\right)$$

$$+ 33 s^4 (1-c) \sin\left(6f + 6g - h + \theta_6^{(1)}\right)\Big\}$$

$$+ 2.5512403 \times 10^{-2} \frac{\mu \bar{J}_6^{(2)} a_e^6}{r^7} \left\{ 20 s^2 (1 - 18 c^2 + 33 c^4) \cos 2 \left(h - \theta_6^{(2)} \right) \right.$$

$$- (1 + c)^2 (17 - 108 c - 90 c^2 + 660 c^3 - 495 c^4) \cdot$$

$$\cdot \cos 2 \left(f + g + h - \theta_6^{(2)} \right)$$

$$- (1 - c)^2 (17 + 108 c - 90 c^2 - 660 c^3 - 495 c^4) \cdot$$

$$\cdot \cos 2 \left(f + g - h + \theta_6^{(2)} \right)$$

$$+ 6 s^2 (1 + c)^2 (1 - 22 c + 33 c^2) \cos \left(4 f + 4 g + 2 h - 2 \theta_6^{(2)} \right)$$

$$+ 6 s^2 (1 - c)^2 (1 + 22 c + 33 c^2) \cos \left(4 f + 4 g - 2 h + 2 \theta_6^{(2)} \right)$$

$$+ 33 s^4 (1 + c)^2 \cos \left(6 f + 6 g + 2 h - 2 \theta_6^{(2)} \right)$$

$$\left. + 33 s^4 (1 - c)^2 \cos \left(6 f + 6 g - 2 h + 2 \theta_6^{(2)} \right) \right\}$$

$$+ 5.1024807 \times 10^{-2} \frac{\mu \bar{J}_6^{(3)} a_e^6}{r^7} \, s \left\{ - 20 s^2 c (3 - 11 c^2) \sin 3 \left(h - \theta_6^{(3)} \right) \right.$$

$$+ 3 (1 + c)^2 (3 + 5c - 55c^2 + 55c^3) \sin \left(2f + 2g + 3h - 3\theta_6^{(3)} \right)$$

$$+ 3 (1 - c)^2 (3 - 5c - 55c^2 - 55c^3) \sin \left(2f + 2g - 3h + 3\theta_6^{(3)} \right)$$

$$- 6 (1 + c)^3 (2 - 11 c + 11 c^2) \sin \left(4 f + 4 g + 3 h - 3 \theta_6^{(3)} \right)$$

$$- 6 (1 - c)^3 (2 + 11 c + 11 c^2) \sin \left(4 f + 4 g - 3 h + 3 \theta_6^{(3)} \right)$$

$$- 11 s^2 (1 + c)^3 \sin \left(6 f + 6 g + 3 h - 3 \theta_6^{(3)} \right)$$

$$\left. - 11 s^2 (1 - c)^3 \sin \left(6 f + 6 g - 3 h + 3 \theta_6^{(3)} \right) \right\}$$

$$+\,2.7947438\times 10^{-2}\,\frac{\mu \bar{J}_6^{(4)} a_e^6}{r^7}\left\{20s^4(1-11c^2)\cos 4\left(h-\theta_6^{(4)}\right)\right.$$

$$-\,5s^2(1+c)^2(1-22c+33c^2)\cos\left(2f+2g+4h-4\theta_6^{(4)}\right)$$

$$-\,5s^2(1-c)^2(1+22c+33c^2)\cos\left(2f+2g-4h+4\theta_6^{(4)}\right)$$

$$-\,2(1+c)^4(13-44c+33c^2)\cos\left(4f+4g+4h-4\theta_6^{(4)}\right)$$

$$-\,2(1-c)^4(13+44c+33c^2)\cos\left(4f+4g-4h+4\theta_6^{(4)}\right)$$

$$-\,11s^2(1+c)^4\cos\left(6f+6g+4h-4\theta_6^{(4)}\right)$$

$$\left.-\,11s^2(1-c)^4\cos\left(6f+6g-4h+4\theta_6^{(4)}\right)\right\}$$

$$+\,.13108510\,\frac{\mu \bar{J}_6^{(5)} a_e^6}{r^7}\;s\left\{-\,20s^4\,c\,\sin 5\left(h-\theta_6^{(5)}\right)\right.$$

$$+\,5s^2(1+c)^2(1-3c)\sin\left(2f+2g+5h-5\theta_6^{(5)}\right)$$

$$+\,5s^2(1-c)^2(1+3c)\sin\left(2f+2g-5h+5\theta_6^{(5)}\right)$$

$$+\,2(1+c)^4(2-3c)\sin\left(4f+4g+5h-5\theta_6^{(5)}\right)$$

$$+\,2(1-c)^4(2+3c)\sin\left(4f+4g-5h+5\theta_6^{(5)}\right)$$

$$+\,(1+c)^5\sin\left(6f+6g+5h-5\theta_6^{(5)}\right)$$

$$\left.+\,(1-c)^5\sin\left(6f+6g-5h+5\theta_6^{(5)}\right)\right\}$$

$$+\ 3.7841009 \times 10^{-2} \frac{\mu \bar{J}_6^{(6)} a_e^6}{r^7} \Big\{ 20 s^6 \cos 6\left(h - \theta_6^{(6)}\right)$$

$$+ 15 s^4 (1+c)^2 \cos\left(2f + 2g + 6h - 6\theta_6^{(6)}\right)$$

$$+ 15 s^4 (1-c)^2 \cos\left(2f + 2g - 6h + 6\theta_6^{(6)}\right)$$

$$+ 6 s^2 (1+c)^4 \cos\left(4f + 4g + 6h - 6\theta_6^{(6)}\right)$$

$$+ 6 s^2 (1-c)^4 \cos\left(4f + 4g - 6h + 6\theta_6^{(6)}\right)$$

$$+ (1+c)^6 \cos 6\left(f + g + h - \theta_6^{(6)}\right)$$

$$+ (1-c)^6 \cos 6\left(f + g - h + \theta_6^{(6)}\right) \Big\}$$

$$+\ 2.5016970 \times 10^{-3} \frac{\mu \bar{J}_7^{(1)} a_e^7}{r^8} \Big\{ 5(1+c)(5 + 270c - 405c^2 - 1980c^3 + 2475c^4 + 2574c^5$$

$$- 3003c^6) \cdot \cos\left(f + g + h - \theta_7^{(1)}\right)$$

$$+ 5(1-c)(5 - 270c - 405c^2 + 1980c^3 + 2475c^4 - 2574c^5$$

$$- 3003c^6) \cdot \cos\left(f + g - h + \theta_7^{(1)}\right)$$

$$- 9 s^2 (1+c)(9 + 132c - 330c^2 - 572c^3 + 1001c^4) \cdot$$

$$\cdot \cos\left(3f + 3g + h - \theta_7^{(1)}\right)$$

$$- 9 s^2 (1-c)(9 - 132c - 330c^2 + 572c^3 + 1001c^4) \cdot$$

$$\cdot \cos\left(3f + 3g - h + \theta_7^{(1)}\right)$$

$$+ 33 s^4 (1+c)(5 + 26c - 91c^2) \cos\left(5f + 5g + h - \theta_7^{(1)}\right)$$

$$+ 33 s^4 (1-c)(5 - 26c - 91c^2) \cos\left(5f + 5g - h + \theta_7^{(1)}\right)$$

$$- 429s^6(1+c)\cos\left(7f+7g+h-\theta_7^{(1)}\right)$$

$$- 429s^6(1-c)\cos\left(7f+7g-h+\theta_7^{(1)}\right)\Big\}$$

$$+ 6.1278811 \times 10^{-3} \frac{\mu \bar{J}_7^{(2)} a_e^7}{r^8} \; s \Big\{ 5(1+c)(15 - 45c - 330c^2 + 550c^3 + 715c^4 - 1001c^5) \cdot$$

$$\cdot \sin\left(f+g+2h-2\theta_7^{(2)}\right)$$

$$+ 5(1-c)(15 + 45c - 330c^2 - 550c^3 + 715c^4 + 1001c^5) \cdot$$

$$\cdot \sin\left(f+g-2h+2\theta_7^{(2)}\right)$$

$$- 3(1+c)^2(19 - 176c - 66c^2 + 1144c^3 - 1001c^4) \cdot$$

$$\cdot \sin\left(3f+3g+2h-2\theta_7^{(2)}\right)$$

$$- 3(1-c)^2(19 + 176c - 66c^2 - 1144c^3 - 1001c^4) \cdot$$

$$\cdot \sin\left(3f+3g-2h+2\theta_7^{(2)}\right)$$

$$+ 11s^2(1+c)^2(1 - 52c + 91c^2)\sin\left(5f+5g+2h-2\theta_7^{(2)}\right)$$

$$+ 11s^2(1-c)^2(1 + 52c + 91c^2)\sin\left(5f+5g-2h+2\theta_7^{(2)}\right)$$

$$+ 143s^4(1+c)^2\sin\left(7f+7g+2h-2\theta_7^{(2)}\right)$$

$$+ 143s^4(1-c)^2\sin\left(7f+7g-2h+2\theta_7^{(2)}\right)\Big\}$$

$$+ 4.3330662 \times 10^{-3} \frac{\mu \bar{J}_7^{(3)} a_e^7}{r^8} \Big\{ 5s^2(1+c)(9 + 132c - 330c^2 - 572c^3 + 1001c^4) \cdot$$

$$\cdot \cos\left(f+g+3h-3\theta_7^{(3)}\right)$$

$$+ 5s^2(1 - c)(9 - 132c - 330c^2 + 572c^3 + 1001c^4)\cdot$$

$$\cdot \cos\left(f + g - 3h + 3\theta_7^{(3)}\right)$$

$$- 3(1 + c)^3(39 - 44c - 726c^2 + 1716c^3 - 1001c^4)\cdot$$

$$\cdot \cos 3\left(f + g + h - \theta_7^{(3)}\right)$$

$$- 3(1 - c)^3(39 + 44c - 726c^2 - 1716c^3 - 1001c^4)\cdot$$

$$\cdot \cos 3\left(f + g - h + \theta_7^{(3)}\right)$$

$$+ 11s^2(1 + c)^3(11 - 78c + 91c^2)\cos\left(5f + 5g + 3h - 3\theta_7^{(3)}\right)$$

$$+ 11s^2(1 - c)^3(11 + 78c + 91c^2)\cos\left(5f + 5g - 3h + 3\theta_7^{(3)}\right)$$

$$+ 143s^4(1 + c)^3 \cos\left(7f + 7g + 3h - 3\theta_7^{(3)}\right)$$

$$+ 143s^4(1 - c)^3 \cos\left(7f + 7g - 3h + 3\theta_7^{(3)}\right)\Bigg\}$$

$$+ 2.8742310 \times 10^{-2} \frac{\mu \bar{J}_7^{(4)} a_e^7}{r^8}\ s \Bigg\{5s^2(1 + c)(3 - 15c - 39c^2 + 91c^3)\sin\left(f + g + 4h - 4\theta_7^{(4)}\right)$$

$$+ 5s^2(1 - c)(3 + 15c - 39c^2 - 91c^3)\sin\left(f + g - 4h + 4\theta_7^{(4)}\right)$$

$$+ 3(1 + c)^3(1 + 33c - 117c^2 + 91c^3)\sin\left(3f + 3g + 4h - 4\theta_7^{(4)}\right)$$

$$+ 3(1 - c)^3(1 - 33c - 117c^2 - 91c^3)\sin\left(3f + 3g - 4h + 4\theta_7^{(4)}\right)$$

$$- (1 + c)^4(25 - 104c + 91c^2)\sin\left(5f + 5g + 4h - 4\theta_7^{(4)}\right)$$

$$- (1 - c)^4(25 + 104c + 91c^2)\sin\left(5f + 5g - 4h + 4\theta_7^{(4)}\right)$$

$$- 13s^2(1+c)^4 \sin\left(7f + 7g + 4h - 4\theta_7^{(4)}\right)$$

$$- 13s^2(1-c)^4 \sin\left(7f + 7g - 4h + 4\theta_7^{(4)}\right)\Big\}$$

$$+ 1.4371155 \times 10^{-2} \frac{\mu \bar{J}_7^{(5)} a_e^7}{r^8} \Big\{5s^4(1+c)(5+26c-91c^2)\cos\left(f+g+5h-5\theta_7^{(5)}\right)$$

$$+ 5s^4(1-c)(5-26c-91c^2)\cos\left(f+g-5h+5\theta_7^{(5)}\right)$$

$$- 3s^2(1+c)^3(11-78c+91c^2)\cos\left(3f+3g+5h-5\theta_7^{(5)}\right)$$

$$- 3s^2(1-c)^3(11+78c+91c^2)\cos\left(3f+3g-5h+5\theta_7^{(5)}\right)$$

$$- (1+c)^5(43-130c+91c^2)\cos 5\left(f+g+h-\theta_7^{(5)}\right)$$

$$- (1-c)^5(43+130c+91c^2)\cos 5\left(f+g-h+\theta_7^{(5)}\right)$$

$$- 13s^2(1+c)^5 \cos\left(7f + 7g + 5h - 5\theta_7^{(5)}\right)$$

$$- 13s^2(1-c)^5 \cos\left(7f + 7g - 5h + 5\theta_7^{(5)}\right)\Big\}$$

$$+ 7.3278799 \times 10^{-2} \frac{\mu \bar{J}_7^{(6)} a_e^7}{r^8}\, s \Big\{5s^4(1+c)(1-7c)\sin\left(f+g+6h-6\theta_7^{(6)}\right)$$

$$+ 5s^4(1-c)(1+7c)\sin\left(f+g-6h+6\theta_7^{(6)}\right)$$

$$+ 3s^2(1+c)^3(3-7c)\sin\left(3f+3g+6h-6\theta_7^{(6)}\right)$$

$$+ 3s^2(1-c)^3(3+7c)\sin\left(3f+3g-6h+6\theta_7^{(6)}\right)$$

$$+ (1+c)^5(5-7c)\sin\left(5f+5g+6h-6\theta_7^{(6)}\right)$$

$$+ (1-c)^5(5+7c)\sin\left(5f+5g-6h+6\theta_7^{(6)}\right)$$

$$+ (1+c)^6 \sin\left(7f + 7g + 6h - 6\theta_7^{(6)}\right)$$

$$+ (1-c)^6 \sin\left(7f + 7g - 6h + 6\theta_7^{(6)}\right)\Big\}$$

$$+ 3.5313941 \times 10^{-4} \frac{\mu \bar{J}_{13}^{(13)} a_e^{13}}{r^{14}} \Big\{1716 s^{12} (1+c) \cos\left(f + g + 13h - 13\theta_{13}^{(13)}\right)$$

$$+ 1716 s^{12} (1-c) \cos\left(f + g - 13h + 13\theta_{13}^{(13)}\right)$$

$$+ 1287 s^{10} (1+c)^3 \cos\left(3f + 3g + 13h - 13\theta_{13}^{(13)}\right)$$

$$+ 1287 s^{10} (1-c)^3 \cos\left(3f + 3g - 13h + 13\theta_{13}^{(13)}\right)$$

$$+ 715 s^8 (1+c)^5 \cos\left(5f + 5g + 13h - 13\theta_{13}^{(13)}\right)$$

$$+ 715 s^8 (1-c)^5 \cos\left(5f + 5g - 13h + 13\theta_{13}^{(13)}\right)$$

$$+ 286 s^6 (1+c)^7 \cos\left(7f + 7g + 13h - 13\theta_{13}^{(13)}\right)$$

$$+ 286 s^6 (1-c)^7 \cos\left(7f + 7g - 13h + 13\theta_{13}^{(13)}\right)$$

$$+ 78 s^4 (1+c)^9 \cos\left(9f + 9g + 13h - 13\theta_{13}^{(13)}\right)$$

$$+ 78 s^4 (1-c)^9 \cos\left(9f + 9g - 13h + 13\theta_{13}^{(13)}\right)$$

$$+ 13 s^2 (1+c)^{11} \cos\left(11f + 11g + 13h - 13\theta_{13}^{(13)}\right)$$

$$+ 13 s^2 (1-c)^{11} \cos\left(11f + 11g - 13h + 13\theta_{13}^{(13)}\right)$$

$$+ (1+c)^{13} \cos 13\left(f + g + h - \theta_{13}^{(13)}\right)$$

$$+ (1-c)^{13} \cos 13\left(f + g - h + \theta_{13}^{(13)}\right)\Big\}$$

$$+6.5686033\times 10^{-5}\,\frac{\mu \bar{J}_{15}^{(13)} a_e^{15}}{r^{16}}\Big\{429s^{12}(1+c)(13+58c-435c^2)\cos\left(f+g+13h-13\theta_{15}^{(13)}\right)$$

$$+429s^{12}(1-c)(13-58c-435c^2)\cos\left(f+g-13h+13\theta_{15}^{(13)}\right)$$

$$-1001s^{10}(1+c)^3(1-58c+145c^2)\cdot$$

$$\cdot\cos\left(3f+3g+13h-13\theta_{15}^{(13)}\right)$$

$$-1001s^{10}(1-c)^3(1+58c+145c^2)\cdot$$

$$\cdot\cos\left(3f+3g-13h+13\theta_{15}^{(13)}\right)$$

$$-1001s^{8}(1+c)^5(7-58c+87c^2)\cdot$$

$$\cdot\cos\left(5f+5g+13h-13\theta_{15}^{(13)}\right)$$

$$-1001s^{8}(1-c)^5(7+58c+87c^2)\cdot$$

$$\cdot\cos\left(5f+5g-13h+13\theta_{15}^{(13)}\right)$$

$$-91s^{6}(1+c)^7(83-406c+435c^2)\cdot$$

$$\cdot\cos\left(7f+7g+13h-13\theta_{15}^{(13)}\right)$$

$$-91s^{6}(1-c)^7(83+406c+435c^2)\cdot$$

$$\cdot\cos\left(7f+7g-13h+13\theta_{15}^{(13)}\right)$$

$$-91s^{4}(1+c)^9(49-174c+145c^2)\cdot$$

$$\cdot\cos\left(9f+9g+13h-13\theta_{15}^{(13)}\right)$$

$$-91s^4(1-c)^9(49+174c+145c^2)\cdot$$

$$\cdot\cos\left(9f+9g-13h+13\theta_{15}^{(13)}\right)$$

$$-7s^2(1+c)^{11}(227-638c+435c^2)\cdot$$

$$\cdot\cos\left(11f+11g+13h-13\theta_{15}^{(13)}\right)$$

$$-7s^2(1-c)^{11}(227+638c+435c^2)\cdot$$

$$\cdot\cos\left(11f+11g-13h+13\theta_{15}^{(13)}\right)$$

$$-(1+c)^{13}(323-754c+435c^2)\cos 13\left(f+g+h-\theta_{15}^{(13)}\right)$$

$$-(1-c)^{13}(323+754c+435c^2)\cos 13\left(f+g-h+\theta_{15}^{(13)}\right)$$

$$-29s^2(1+c)^{13}\cos\left(15f+15g+13h-13\theta_{15}^{(13)}\right)$$

$$-29s^2(1-c)^{13}\cos\left(15f+15g-13h+13\theta_{15}^{(13)}\right)\Big\}$$

$$+5.0024992\times 10^{-4}\frac{\mu J_{15}^{(14)}a_e^{15}}{r^{16}}\; s\Big\{429s^{12}(1+c)(1-15c)\sin\left(f+g+14h-14\theta_{15}^{(14)}\right)$$

$$+429s^{12}(1-c)(1+15c)\sin\left(f+g-14h+14\theta_{15}^{(14)}\right)$$

$$+1001s^{10}(1+c)^3(1-5c)\sin\left(3f+3g+14h-14\theta_{15}^{(14)}\right)$$

$$+1001s^{10}(1-c)^3(1+5c)\sin\left(3f+3g-14h+14\theta_{15}^{(14)}\right)$$

$$+1001s^8(1+c)^5(1-3c)\sin\left(5f+5g+14h-14\theta_{15}^{(14)}\right)$$

$$+1001s^8(1-c)^5(1+3c)\sin\left(5f+5g-14h+14\theta_{15}^{(14)}\right)$$

$$+91s^6(1+c)^7(7-15c)\sin\left(7f+7g+14h-14\theta_{15}^{(14)}\right)$$

$$+91s^6(1-c)^7(7+15c)\sin\left(7f+7g-14h+14\theta_{15}^{(14)}\right)$$

$$+91s^4(1+c)^9(3-5c)\sin\left(9f+9g+14h-14\theta_{15}^{(14)}\right)$$

$$+91s^4(1-c)^9(3+5c)\sin\left(9f+9g-14h+14\theta_{15}^{(14)}\right)$$

$$+7s^2(1+c)^{11}(11-15c)\sin\left(11f+11g+14h-14\theta_{15}^{(14)}\right)$$

$$+7s^2(1-c)^{11}(11+15c)\sin\left(11f+11g-14h+14\theta_{15}^{(14)}\right)$$

$$+(1+c)^{13}(13-15c)\sin\left(13f+13g+14h-14\theta_{15}^{(14)}\right)$$

$$+(1-c)^{13}(13+15c)\sin\left(13f+13g-14h+14\theta_{15}^{(14)}\right)$$

$$+(1+c)^{14}\sin\left(15f+15g+14h-14\theta_{15}^{(14)}\right)$$

$$+(1-c)^{14}\sin\left(15f+15g-14h+14\theta_{15}^{(14)}\right)\Bigg\}$$

where

$$s = \sin i$$

$$c = \cos i.$$

THE DISTURBING FUNCTION AS A FUNCTION
OF THE RECTANGULAR COORDINATES

If one makes use of the following relationships between the sperical coordinates and the rectangular coordinates of a satellite

$$\sin \phi = \frac{z}{r}$$

$$\cos^m \phi \cos m\lambda = \frac{1}{r^m}\left[x^m - \binom{m}{2} x^{m-2} y^2 + \cdots + (-1)^{\frac{m-2}{2}} \binom{m}{m-2} x^2 y^{m-2} + (-1)^{\frac{m}{2}} y^m\right] \quad (m \text{ even})$$

$$\cos^m \phi \cos m\lambda = \frac{1}{r^m}\left[x^m - \binom{m}{2} x^{m-2} y^2 + \cdots + (-1)^{\frac{m-3}{2}} \binom{m}{m-3} x^3 y^{m-3} + (-1)^{\frac{m-1}{2}} x y^{m-1}\right] \quad (m \text{ odd})$$

$$\cos^m \phi \sin m\lambda = \frac{1}{r^m}\left[\binom{m}{1} x^{m-1} y - \binom{m}{3} x^{m-3} y^3 + \cdots + (-1)^{\frac{m-4}{2}} \binom{m}{m-3} x^3 y^{m-3} + (-1)^{\frac{m-2}{2}} \binom{m}{m-1} x y^{m-1}\right] \quad (m \text{ even})$$

$$\cos^m \phi \sin m\lambda = \frac{1}{r^m}\left[\binom{m}{1} x^{m-1} y - \binom{m}{3} x^{m-3} y^3 + \cdots + (-1)^{\frac{m-3}{2}} \binom{m}{m-2} x^2 y^{m-2} + (-1)^{\frac{m-1}{2}} y^m\right], \quad (m \text{ odd})$$

then the disturbing function as a function of the rectangular coordinates is

$$R = 1.6056541 \frac{\mu a_e^5}{r^{11}} (x^4 + y^4 + 8z^4 + 2x^2y^2 - 12x^2z^2 - 12y^2z^2)\left(\overline{C}_5^{(1)} x + \overline{S}_5^{(1)} y\right)$$

$$+8.4963227 \frac{\mu z a_e^5}{r^{11}} (2z^2-x^2-y^2)\left[\bar{C}_5^{(2)}(x^2-y^2)+2\bar{S}_5^{(2)}xy\right]$$

$$+1.7343046 \frac{\mu a_e^5}{r^{11}} (8z^2-x^2-y^2)\left[\bar{C}_5^{(3)}(x^3-3xy^2)+\bar{S}_5^{(3)}(3x^2y-y^3)\right]$$

$$+7.3580313 \frac{\mu z a_e^5}{r^{11}} \left[\bar{C}_5^{(4)}(x^4-6x^2y^2+y^4)+4\bar{S}_5^{(4)}(x^3y-xy^3)\right]$$

$$+2.3268138 \frac{\mu a_e^5}{r^{11}} \left[\bar{C}_5^{(5)}(x^5-10x^3y^2+5xy^4)+\bar{S}_5^{(5)}(5x^4y-10x^2y^3+y^5)\right]$$

$$+2.0653390 \frac{\mu z a_e^6}{r^{13}} (5x^4+5y^4+8z^4+10x^2y^2-20x^2z^2-20y^2z^2)\left(\bar{C}_6^{(1)}x+\bar{S}_6^{(1)}y\right)$$

$$+1.6327938 \frac{\mu a_e^6}{r^{13}} (x^4+y^4+16z^4+2x^2y^2-16x^2z^2-16y^2z^2)\cdot$$

$$\cdot\left[\bar{C}_6^{(2)}(x^2-y^2)+2\bar{S}_6^{(2)}xy\right]$$

$$+3.2655876 \frac{\mu z a_e^6}{r^{13}} (8z^2-3x^2-3y^2)\left[\bar{C}_6^{(3)}(x^3-3xy^2)+\bar{S}_6^{(3)}(3x^2y-y^3)\right]$$

$$+1.7886360 \frac{\mu a_e^6}{r^{13}} (10z^2-x^2-y^2)\left[\bar{C}_6^{(4)}(x^4-6x^2y^2+y^4)+4\bar{S}_6^{(4)}(x^3y-xy^3)\right]$$

$$+8.3894465 \frac{\mu z a_e^6}{r^{13}} \left[\bar{C}_6^{(5)}(x^5-10x^3y^2+5xy^4)+\bar{S}_6^{(5)}(5x^4y-10x^2y^3+y^5)\right]$$

$$+2.4218246 \frac{\mu a_e^6}{r^{13}} \left[\bar{C}_6^{(6)}(x^6-15x^4y^2+15x^2y^4-y^6)+\bar{S}_6^{(6)}(6x^5y-20x^3y^3+6xy^5)\right]$$

$$+.32021721 \frac{\mu a_e^7}{r^{15}} (64z^6-5x^6-5y^6-15x^2y^4-15x^4y^2-240x^2z^4+120x^4z^2-240y^2z^4$$

$$+120y^4z^2+240x^2y^2z^2)\cdot\left(\bar{C}_7^{(1)}x+\bar{S}_7^{(1)}y\right)$$

$$+ .78436878 \frac{\mu z a_e^7}{r^{15}} (48 z^4 + 15 x^4 + 15 y^4 + 30 x^2 y^2 - 80 x^2 z^2 - 80 y^2 z^2) \cdot$$

$$\cdot \left[\bar{C}_7^{(2)} (x^2 - y^2) + 2 \bar{S}_7^{(2)} x y \right]$$

$$+ .55463248 \frac{\mu a_e^7}{r^{15}} (80 z^4 + 3 x^4 + 3 y^4 + 6 x^2 y^2 - 60 x^2 z^2 - 60 y^2 z^2) \cdot$$

$$\cdot \left[\bar{C}_7^{(3)} (x^3 - 3 x y^2) + \bar{S}_7^{(3)} (3 x^2 y - y^3) \right]$$

$$+ 3.6790157 \frac{\mu z a_e^7}{r^{15}} (10 z^2 - 3 x^2 - 3 y^2) \left[\bar{C}_7^{(4)} (x^4 - 6 x^2 y^2 + y^4) + 4 \bar{S}_7^{(4)} (x^3 y - x y^3) \right]$$

$$+ 1.8395078 \frac{\mu a_e^7}{r^{15}} (12 z^2 - x^2 - y^2) \left[\bar{C}_7^{(5)} (x^5 - 10 x^3 y^2 + 5 x y^4) \right.$$

$$\left. + \bar{S}_7^{(5)} (5 x^4 y - 10 x^2 y^3 + y^5) \right]$$

$$+ 9.3796863 \frac{\mu z a_e^7}{r^{15}} \left[\bar{C}_7^{(6)} (x^6 - 15 x^4 y^2 + 15 x^2 y^4 - y^6) + \bar{S}_7^{(6)} (6 x^5 y - 20 x^3 y^3 + 6 x y^5) \right]$$

$$+ 2.8929181 \frac{\mu a_e^{13}}{r^{27}} \left[\bar{C}_{13}^{(13)} (x^{13} - 78 x^{11} y^2 + 715 x^9 y^4 - 1716 x^7 y^6 + 1287 x^5 y^8 \right.$$

$$- 286 x^3 y^{10} + 13 x y^{12}) + \bar{S}_{13}^{(13)} (13 x^{12} y - 286 x^{10} y^3 + 1287 x^8 y^5 - 1716 x^6 y^7$$

$$\left. + 715 x^4 y^9 - 78 x^2 y^{11} + y^{13}) \right]$$

$$+2.1523999\,\frac{\mu a_e^{15}}{r^{31}}\,(28z^2-x^2-y^2)\Big[\bar{C}_{15}^{(13)}(x^{13}-78x^{11}y^2+715x^9y^4-1716x^7y^6$$

$$+1287x^5y^8-286x^3y^{10}+13xy^{12})+\bar{S}_{15}^{(13)}(13x^{12}y-286x^{10}y^3+1287x^8y^5$$

$$-1716x^6y^7+715x^4y^9-78x^2y^{11}+y^{13})\Big]$$

$$+16.392189\,\frac{\mu z\, a_e^{15}}{r^{31}}\Big[\bar{C}_{15}^{(14)}(x^{14}-91x^{12}y^2+1001x^{10}y^4-3003x^8y^6$$

$$+3003x^6y^8-1001x^4y^{10}+91x^2y^{12}-y^{14})+\bar{S}_{15}^{(14)}(14x^{13}y-364x^{11}y^3+2002x^9y^5$$

$$-3432x^7y^7+2002x^5y^9-364x^3y^{11}+14xy^{13})\Big].$$

UNNORMALIZED COEFFICIENTS

The disturbing functions given above have been written in terms of the normalized coefficients $\bar{J}_n^{(m)}$ and $\bar{C}_n^{(m)}$ and $\bar{S}_n^{(m)}$. It is also possible to obtain these disturbing functions in terms of the unnormalized coefficients. To do this for the disturbing function as a function of the rectangular coordinates, simply remove the bars over the $\bar{C}_n^{(m)}$ and $\bar{S}_n^{(m)}$ and replace the truncated decimal of the (m,n) term by the rational number indicated in Table I.

To find the unnormalized disturbing function written as a function of the orbital elements simply replace the $\bar{J}_n^{(m)}$ by $J_n^{(m)}$ and replace the truncated decimal of the (m,n) term by $1/2^n$ times the rational number indicated in Table I.

Table I

m, n		m, n	
1 5	$\frac{15}{8}$	6 6	10395
2 5	$\frac{105}{2}$	1 7	$\frac{7}{16}$
3 5	$\frac{105}{2}$	2 7	$\frac{63}{8}$
4 5	945	3 7	$\frac{315}{8}$
5 5	945	4 7	$\frac{3465}{2}$
1 6	$\frac{21}{8}$	5 7	$\frac{10395}{2}$
2 6	$\frac{105}{8}$	6 7	135135
3 6	$\frac{315}{2}$	13 13	$(n+m-1)!!$
4 6	$\frac{945}{2}$	13 15	$\frac{1}{2}(n+m-1)!!$
5 6	10395	14 15	$(n+m)!!$